

QCD FROM THE LATTICE

CHRIS MICHAEL

D. A. M. T. P., University of Liverpool

Liverpool L69 3BX, U. K.

E-mail: C.Michael@liv.ac.uk

ABSTRACT

The lattice gauge theory technique for non-perturbative calculations in QCD is reviewed. The extraction of the continuum limit of lattice results is discussed with particular examples appropriate to hadron spectroscopy (the light hadrons and the glueballs). The determination of the strong coupling constant is presented. The lattice approach to the evaluation of hadronic matrix elements appropriate to weak decays is described.

1. Introduction

The lattice approach to QCD is, in principle, an exact and fully non-perturbative method to extract results. It is an essential component in the comparison of predictions of the Standard Model with experiment. There are limitations at present, of course, and I aim to describe the scope and potential of lattice QCD.

The lattice approach to QCD, pioneered by Wilson, involves assigning the colour fields to the links of a lattice¹. I shall concentrate on the use of a four-dimensional lattice with Euclidean time. Then, using periodic space-time boundaries, the functional integral of the quantum field theory becomes a multi-dimensional integral over the finite number of degrees of freedom. Monte Carlo methods allow this integral to be performed to the required precision. This approach thus allows a completely finite formulation of QCD. Furthermore, the approach is fully non-perturbative. Thus, in principle, it appears that QCD is “solved”.

Even though we have an exact non-perturbative method, it is important to check the method carefully. The most delicate point is to confirm that the limit of zero lattice spacing is under control. This approach to the continuum limit will be the first topic I discuss in detail. As an illustration, I will discuss glueball and light hadron spectra. The glueball case is discussed because it is an intriguing manifestation of a possibility in QCD which transcends the quark model. The light hadron spectrum is discussed to show that lattice methods are quantitative in a case where the experimental data are available.

One quantity which forms a link between perturbative and non-perturbative QCD is the running coupling α . An accurate knowledge of α is essential to explore possible contributions from supersymmetric particles to the unification of electro-weak and strong couplings. In principle α can be determined very accurately by lattice methods. I review the current state of lattice determinations.

As well as checking experimentally known quantities, lattice QCD is relied on to

calibrate the hadronic component of measurements that search for physics beyond the Standard Model. Thus weak decays of hadrons containing strange, charmed or bottom quarks are a source of data to determine the CKM mixing matrix and thence to look for possible interactions which do not agree with the Standard Model. Because the Standard Model is defined in terms of the interaction of quarks whereas the experiments are performed with hadrons, an accurate knowledge of hadronic matrix elements is needed. Lattice QCD can determine such matrix elements directly and can also be used to refine theoretical models for them.

In principle, lattice QCD provides *ab initio* computations of QCD. The only inputs are the quark masses and one dimensionful observable to set the scale. In practice, computational limitations arise. One of the main limitations is in dealing with light quarks. Consider a quark of mass m_q . Fermions are treated by integrating out the quark contributions analytically which yields a large sparse fermionic matrix M . The computation then involves estimation of the inverse of this matrix M . For heavy quarks, this inverse can be evaluated reliably and quickly by iteration. For very light quarks (ie with physical masses of u and d quarks), this iteration is unreliable. The computational strategy is then to evaluate quantities for a range of quark masses m_q (corresponding to the strange quark mass and heavier) and to extrapolate to the required light quark masses.

Because of this limitation of dealing with light quarks, a very popular compromise is to use the “quenched approximation”. In this approximation, the sea-quark mass is taken as very heavy so that quark loops in the vacuum are neglected. The valence quark masses are treated as described above by extrapolation to the required light quark mass value. As I shall describe, the quenched approximation seems to reproduce fairly well the light quark spectrum and is thus a useful model for QCD.

One may go one step beyond the quenched approximation by explicitly evaluating the contribution of a quark loop. This enables studies ² of topics such as the η , η' system, s -quarks in the nucleon and glueball decay.

To go to full QCD, the sea-quark mass must be taken finite and an extrapolation must be made to physical light quark masses. Because of the computational overhead of this study, only limited exploration has been made: for example the lattice spacing is usually kept rather coarse. Some relevant results will be discussed. This is an area where future progress is needed.

1.1. The Continuum Limit

Lattice computations use a finite lattice spacing (conventionally called a). We require a to be smaller than the overall size of a hadron and small enough to reproduce sufficiently high energy fluctuations in the fields. Current lattice simulations use a lattice spacing a ranging from 0.07 fm to 0.20 fm. This is a very reasonable range given our knowledge of the size of hadrons (over 0.5 fm) and the energy scale of the

dynamics since distances below 0.2 fm correspond to energies above 1 GeV. However, it is essential to quantify the discretisation error from using a non-zero lattice spacing. I begin by discussing computations in the quenched approximation.

The discretisation error arising from a given lattice formulation can be calculated theoretically. For the simplest discretisation of the gauge action, that proposed by Wilson, this is known to be an order a^2 contribution to a dimensionless ratio of observables. Thus if a mass ratio is measured for several values of a , a reliable extrapolation can be made to $a = 0$. This is illustrated in fig 1. Clearly an accurate extrapolation to the continuum limit can be made from the combined data shown for the 0^{++} glueball mass. The earliest results obtained were at rather coarse lattice spacings because of computational limitations (ie those on the right on the figure) and taken alone they led to a value of the 0^{++} mass which was too small.

In order to extract a continuum mass, it is appropriate to study a dimensionless ratio to the quantity which is most accurately determined in lattice simulation. The potential $V(R)$ between static quarks is accurately measured on a lattice. Usually the energy scale from the potential is specified by \sqrt{K} where K (or σ) is the string tension given by $\lim_{R \rightarrow \infty} dV/dR$. Because this definition involves an extrapolation in R , it is more precise ⁷ to set the scale by using the potential at finite R and choosing an energy scale r_0^{-1} given implicitly by $R^2 dV/dR|_{r_0} = 1.65$. In practice these scales are closely related since $\sqrt{K}r_0 \approx 1.18$.

In order to give an estimate of the continuum mass in GeV, one needs to specify a value for r_0 (or \sqrt{K}) in GeV. If all predictions of the quenched approximation were to agree with experiment, this would be straightforward. Although many mass ratios determined in the quenched approximation do agree with experiment, since some quantities seem to depart from the experimental ratios by 10 – 20%, it is prudent to retain an overall systematic error of 10% on the energy scale. From the potential model of $b\bar{b}$ mesons, one obtains experimental values for these scales and to be specific I take $\sqrt{K} = 0.44(4)$ GeV which corresponds to $r_0^{-1} = 0.37(4)$ GeV.

1.2. Glueballs

The 0^{++} and 2^{++} glueball masses in terms of r_0 are shown in fig 1. Using the theoretical input that these ratios have corrections of order a^2 , the limits as $a \rightarrow 0$ are rather well determined. Since the errors on individual lattice determinations are usually dominated by systematic errors, some care is needed in fitting the combined results of different groups. A reasonable approach leads to continuum ratios of $m(0^{++})r_0 = 4.3(2)$ and $m(2^{++})r_0 = 6.0(4)$. Then, using the scale determined as discussed above, the lattice glueball data yield 0^{++} and 2^{++} masses of 1.6(2) and 2.2(4) GeV respectively. This systematic error in the energy scale can only be removed by going beyond the quenched approximation. Then, however, glueball–meson mixing will give problems in glueball identification which will be similar to those in experi-

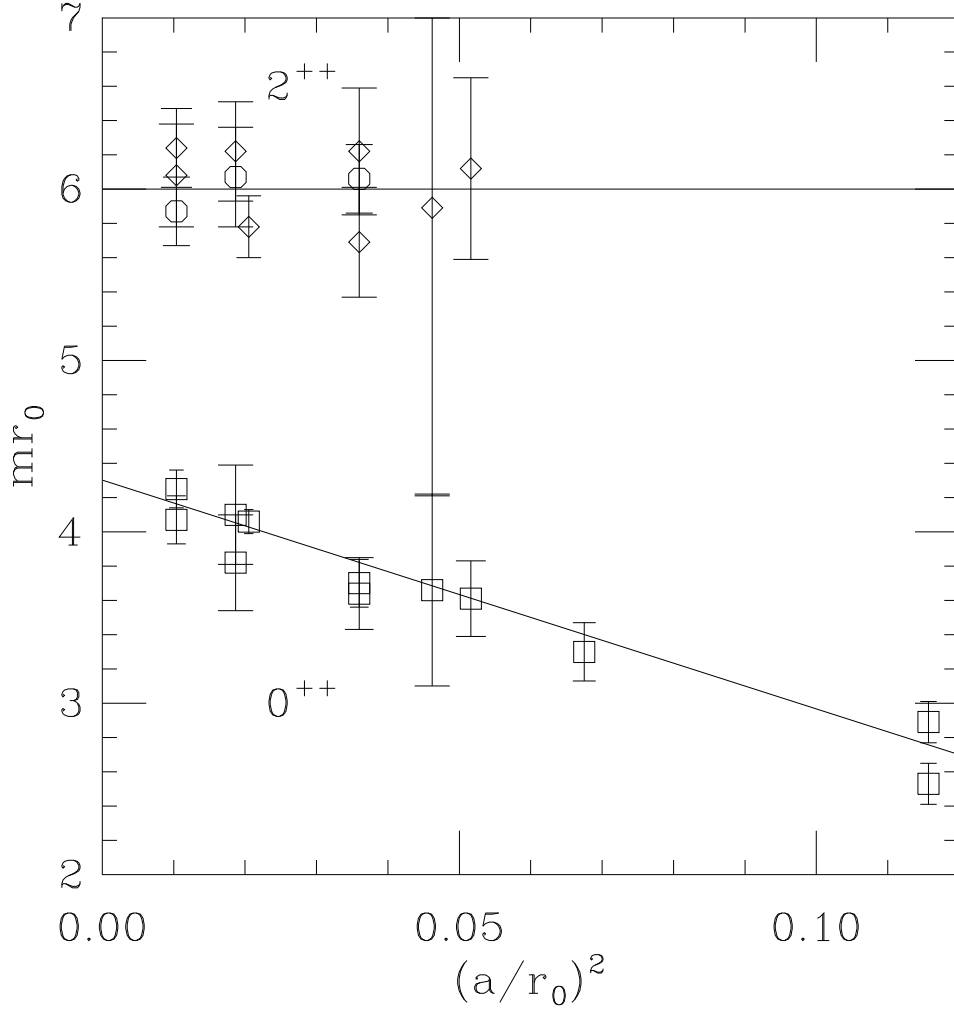


Fig. 1. The mass of the $J^{PC} = 0^{++}$ and 2^{++} glueball states from refs^{3,4,5,6} in units of r_0 . The straight lines show fits describing the approach to the continuum limit as $a \rightarrow 0$.

ments.

The lattice predictions^{4,5} for glueballs of other J^{PC} values are that they lie higher in mass, except for the 0^{-+} state which appears to be at a similar mass to the 2^{++} . There are no low-lying “odd-ball” states with J^{PC} values not allowed by the quark model.

1.3. Light hadron spectrum

There are several alternative formalisms for describing fermions on a lattice and I will show that they agree in the continuum limit. I first discuss the results in the quenched approximation with various valence quark masses m_q . As m_q is decreased, the pseudoscalar meson mass is found to extrapolate to zero while the vector meson

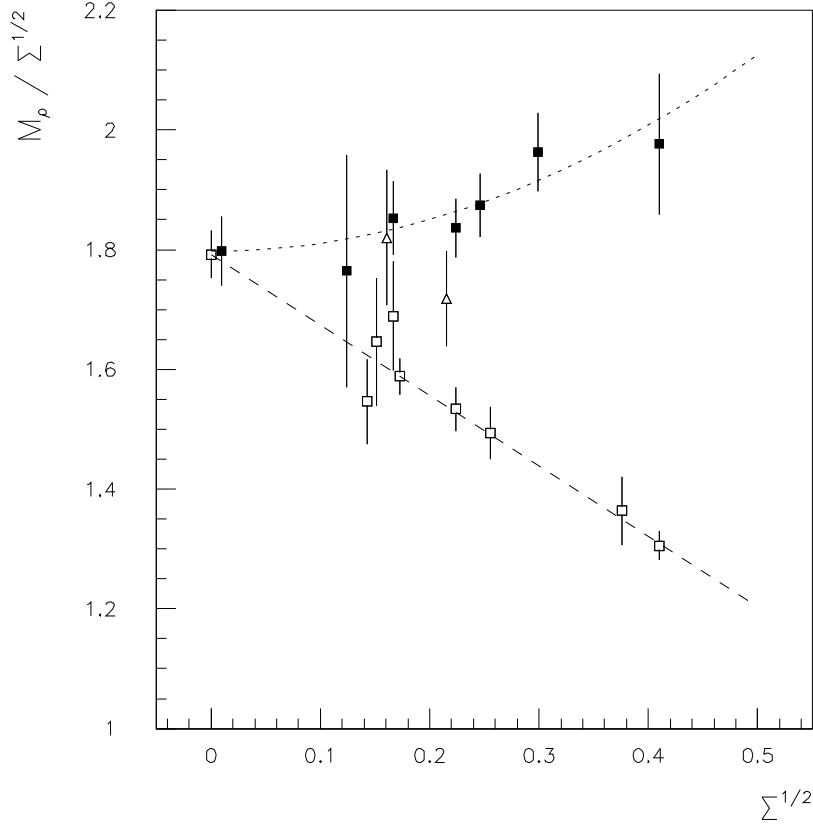


Fig. 2. The mass of the ρ meson as a dimensionless ratio to the lattice string tension $\Sigma = Ka^2$. The continuum limit corresponds to $a \rightarrow 0$ at the left hand side; the right hand side corresponds to $a \approx 0.2$ fm. The quenched lattice data with different fermionic discretisations are compared ⁸. The filled squares are with staggered fermions, the open squares are from Wilson fermions and the triangles are from Clover fermions. The lines illustrate the discretisation errors which behave as a^2 , a and αa respectively.

mass tends to a finite limit: to be identified with the mass of the ρ -meson. Thus the quenched approximation does show spontaneous chiral symmetry breaking.

The dimensionless ratio of the ρ mass to \sqrt{K} is shown by a compilation ⁸ of lattice data in fig 2 versus the lattice spacing a . The dashed and dotted curves show that a common continuum limit exists of $m_\rho/\sqrt{K} = 1.80(5)$ from the different fermionic formulations presented. Moreover, using the experimental values of the string tension (namely $\sqrt{K} = 0.44$ GeV as discussed above) and ρ -meson mass leads to $m_\rho/\sqrt{K} = 1.75$ which is consistent within the errors with the continuum limit of the lattice results. So in this case the quenched ratio agrees with the experimental ratio.

The mass ratio of the proton to ρ is widely known to have a tendency to come out too large in quenched lattice calculations. Sceptics may be unimpressed by getting a value closer to the naive quark model ratio of 1.5 than to the experimental value of

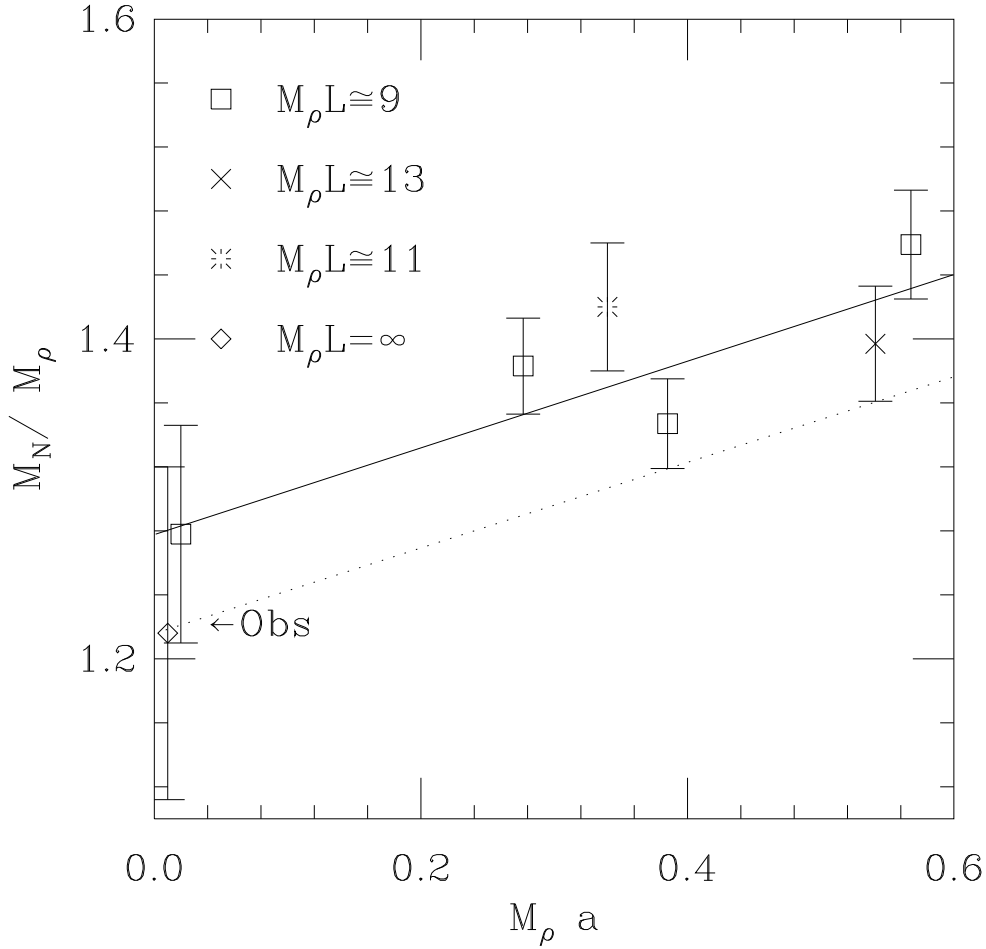


Fig. 3. The nucleon to ρ mass ratio from Wilson fermions in the quenched approximation^{9,10} is plotted versus lattice spacing. The extrapolation to the continuum limit ($a = 0$) is shown as a solid line. The lower dotted curve comes from an additional extrapolation in spatial size L to infinite spatial volume and is shown to agree with the observed ratio (Obs).

1.22. The present situation is that lattice data *are* consistent with the experimental value when an extrapolation to the continuum limit is made: see fig 3. A further extrapolation to infinite spatial volume is also needed and there is some evidence⁹ that this can further improve the agreement. Although the data are consistent with the experimental value, further reduction in the errors of the lattice results is needed to answer definitely whether the quenched approximation reproduces this mass ratio or not.

Studies have been made of the mesons and baryons which are composed of light and strange quarks. The surprise is that the quenched approximation seems to reproduce⁹ these experimental values quite well. This may reflect the relatively large errors that are still present in the lattice determinations. It may also reflect the fact that the hadronic dynamics has a similar energy scale in each case so that the

quenched approximation makes similar errors — which cancel in mass ratios. Some possible discrepancies have been reported, however. One is the dimensionless ratio of $m_{K^*}(m_{K^*} - m_\rho)/(m_K^2 - m_\pi^2)$ which comes out about 20% too small^{12,13} in quenched evaluations. Another is the ratio of $b\bar{b}$ splittings $(1P - 1S)/(2S - 1S)$ which is found to be^{14,16} about 10% low.

2. Running Coupling

The QCD coupling constant α is a key quantity in looking for physics beyond the minimal Standard Model. The perturbative expansion of QCD is accurate for energy scales beyond 10 GeV where the variation (running) of the coupling with energy to even higher energies is given by the perturbative beta-function.

$$\frac{d\alpha}{dq} = -\frac{\beta_0}{2\pi}\alpha^2 - \frac{\beta_1}{8\pi^2}\alpha^3 - \dots$$

Although lattice methods only allow a direct study of the coupling up to scales of order 10 GeV, I shall follow convention and quote values of α at $q = m_Z$ using perturbative evolution (in the $\overline{\text{MS}}$ scheme) to that energy. Different regularisation schemes lead to different values of α . These can be related perturbatively. For instance schemes ‘A’ and ‘B’ are related by the ‘matching’ formula

$$\alpha_A(q) = \alpha_B(q) + c_1\alpha_B^2(q) + c_2\alpha_B^3(q) + \dots$$

where c_1 and c_2 can be calculated perturbatively.

Experimental determinations of α inevitably need modelling of non-perturbative effects since hadrons rather than quarks and gluons are actually observed. Because of this, it is attractive to use the best non-perturbative method available when determining α : namely lattice QCD.

In summaries of determinations of $\alpha_{\overline{\text{MS}}}(m_Z)$, the lattice results have relatively small errors. Several lattice groups have produced results. At Lattice 1993 a value of 0.108(6) was quoted¹⁵ whereas the NRQCD group quoted¹⁶ a value of 0.115(2) at Lattice 1994 and 1995. Recent summaries^{11,17} of lattice determinations gave 0.112(7).

These values are close to the current world average value¹⁸ of 0.117(6).

Because the lattice method is potentially the most accurate, I discuss in some detail the possibilities and limitations of lattice determinations of α .

2.1. Lattice perturbation theory

The lattice formalism specifies the bare gauge coupling constant (since the calculations fix the parameter $\beta = 6/g^2$). So at $\beta = 6.5$, one has $\alpha = g^2/(4\pi) = 0.07346$. This is a small value of the bare coupling and it was expected that the perturbation series derived from the lattice Lagrangian would be well behaved at this β -value.

However, it is now known that the perturbation series in the bare lattice coupling converges very poorly. For instance the series for the logarithm of the plaquette action S_\square has been calculated analytically ²⁰ for $N_f = 0$ to order α^3 for the Wilson gauge action. Indeed the first eight terms of this perturbation series have been evaluated by a numerical technique ¹⁹. For SU(3) colour,

$$\alpha_\square \equiv -\frac{3}{4\pi} \log(S_\square) = \alpha(1 + 3.37\alpha + 17.69\alpha^2 + \dots)$$

Now, for $\alpha = 0.07346$, the right hand expression evaluates to 92% of the left hand expression obtained from the measured lattice plaquette action. The α^3 term contributes 7%. So we do not have a good convergence of perturbation theory. Decreasing the bare α by a substantial factor would imply a reduction of the lattice spacing by a huge factor which is not feasible with present computing techniques while keeping the spatial volume a reasonable physical size. Another way forward is needed.

The root of the poor convergence of lattice perturbation theory is thought to come from tadpole diagrams ²². The way forward has been known for a long time. A *physical definition* of α should yield a perturbative series with coefficients which are much smaller than in the case of the bare coupling. This discussion of improved lattice definitions of couplings is very analogous to the argument that the $\overline{\text{MS}}$ scheme is preferred to the MS scheme in the continuum. Several improved definitions of α have been proposed which are appropriate to a lattice:

- (i) using the potential $V(R)$ between heavy quarks through either α_V determined ²² from $V(q)$ or α_F determined ²¹ from the force dV/dR ,
- (ii) using the logarithm of the plaquette action as a definition ²² of α_\square ,
- (iii) using variation of the boundary conditions yielding α_{SF} from the Schrödinger functional ²³ and α_{TP} from twisted boundary conditions ²⁴.

As an illustration, fig 4 shows that α_F determined at $\beta = 6.5$ from the potential ^{21,25} runs with energy in a way consistent with perturbation theory to 2 loops and has a value which is very different from the bare α of 0.073. Thus the bare lattice coupling is seen to be anomalously small which explains why it is a poor expansion parameter.

What needs to be shown is that different improved definitions of α on a lattice agree with each other when the perturbative matching is used. Direct tests of this are possible using quenched SU(2) where studies using very small lattice spacings and, hence, large energies have been feasible. A comparison of α_{SF} with α_{TP} shows ²⁴ that, provided the scales are set appropriately, the results agree with each other and with two loop perturbative evolution. It is important to calibrate the α_\square determination against other methods since this is the method that will be used in the full QCD results discussed later. The result ²³ is that $\alpha_{SF} = 0.1675(35)$ at $q = 1/a(\beta = 2.85)$ and, since the two-loop matching is known ²⁶, one can use α_\square to evaluate α_{SF} at the same scale obtaining 0.1674(5). This excellent agreement between two different definitions is an indication that perturbation theory on a lattice is now under control.

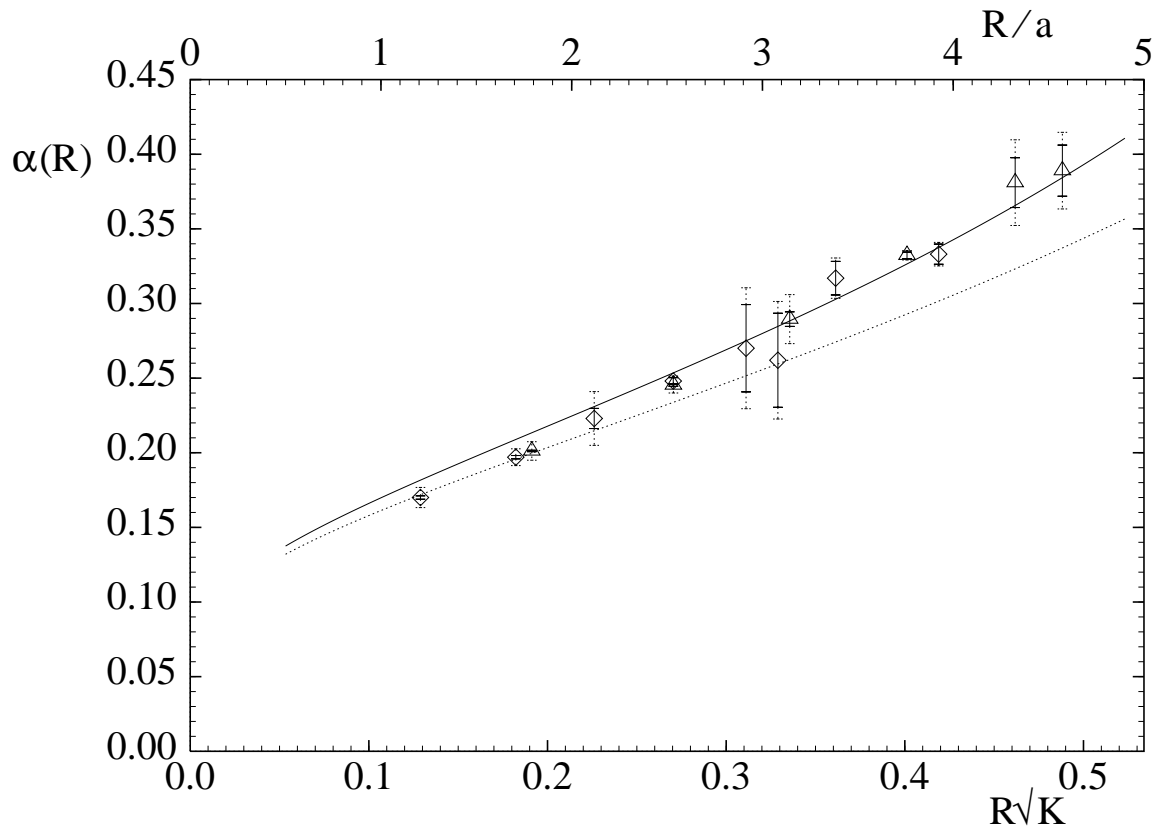


Fig. 4. The effective running coupling constant α_F obtained ²⁵ from the force between static quarks at separation R in the quenched approximation. The scale is set by the string tension K and corresponds to energy $q \approx 4$ GeV to the left and $q \approx 1$ GeV to the right. The curves show the two-loop perturbative evolution for two different values of Λ . The perturbative evolution is not expected to be valid for $q \approx 1$ GeV.

A similar comparison ¹¹ can be made between lattice determinations of α_F , α_{SF} and α_\square for quenched SU(3) and excellent agreement is obtained.

2.2. Lattice determinations of the coupling

It is straightforward in principle to determine α on a lattice: one uses one of the physical definitions to measure α at a scale determined by a non-perturbative measurement of a quantity that is well known experimentally (eg. the ρ mass, charmonium energy splittings or bottomonium energy splittings). It is important to verify that the result is stable as the lattice spacing is reduced.

At present the main limitation to this programme is that full QCD lattice simulations have only been conducted at rather coarse lattice spacing. The NRQCD group overcome this by using an effective Lagrangian which is claimed to be accurate at such

lattice spacings. They extrapolate results at $N_f = 0$ (ie quenched) and $N_f = 2$ to the required $N_f = 3$. They set the scale using several charmonium and bottomonium energy splittings and find consistent results¹⁶ for α_\square . Using one-loop perturbative matching, they obtain $\alpha_{\overline{\text{MS}}} = 0.115(2)$.

A more direct route to α uses Wilson fermions to determine the ρ mass and charmonium energy levels. Again an extrapolation in N_f is needed. The Kyoto-Tsukuba group also make²⁷ an extrapolation in the sea-quark mass to physical sea-quark masses and their final result is $\alpha_{\overline{\text{MS}}} = 0.111(5)$.

Each approach determines the coupling via α_\square which introduces a perturbative error of $\pm 0.002c_2$ from the (as yet) unknown two-loop matching coefficient c_2 . The NRQCD method uses an effective Lagrangian with coefficients obtained by perturbative evaluation, so it is hard to estimate the systematic errors arising from this perturbative approximation. In the more direct approach, the order a effects can result in a scale error of 20% on the lattice which converts into 4% at m_Z . The incomplete treatment of the sea-quarks in both mass and number of flavours also introduces a further systematic error. Thus a conservative summary^{11,17} of current lattice determinations is $\alpha_{\overline{\text{MS}}} = 0.112(7)$.

It should be emphasized that, compared to other methods, the lattice determination of α has no model assumptions. With further work on (i) the two-loop matching, (ii) varying the sea-quark mass, (iii) exploring the N_f dependence and (iv) reducing the lattice spacing a , it should be possible to reduce the error on α to 1-2 %.

3. Hadronic matrix elements

Hadronic matrix elements which can be expressed as Green functions can be determined from first principles using lattice techniques. There is an extensive literature on this subject^{28,8} and I shall select only some topics. Most current studies are in the quenched approximation which introduces a source of systematic error. Even with this quenching error, lattice results are less model-dependent than most other theoretical approaches. Furthermore, it is clear how to remove this quenching error by full QCD lattice studies.

One area of importance phenomenologically is matrix elements of electro-weak operators between hadronic states containing heavy (ie charm or bottom) quarks. These hadronic matrix elements are needed to relate the quark couplings (CKM matrix elements) to the experimental data. They are also needed to establish whether rare decays agree with the Standard Model. Rare decays such as $B \rightarrow K^* \gamma$ involve loops with heavy intermediate particles and are thus prime candidates for tests of the Standard Model.

There are two approaches to studying heavy quarks of mass m_Q on a lattice. One can treat the quarks as infinitely heavy, the static approximation, which is very straightforward to implement since the quarks become effectively just colour sources.

Alternatively, one can use propagating quarks. For propagating quarks, one needs $m_Q a \leq 1$ so that the quarks do not ‘fall through the gaps in the lattice’. This latter requirement currently implies $m_Q \leq 3$ GeV, although an improved fermionic discretisation (such as the SW-clover formalism) should allow $m_Q \approx 3$ GeV.

Since the b -quark lies outside each of these approaches, one needs some theoretical input to bridge the gap. This is provided ²⁹ by the Heavy Quark Effective Theory (HQET) which establishes an expansion in $1/m_Q$ about the static limit. These $1/m_Q$ terms can be explored non-perturbatively on a lattice by combining studies of the quenched approximation with propagating quark results.

A different approach to heavy quarks is provided by the effective Lagrangian of non-relativistic QCD (NRQCD). This has been used successfully for spectroscopy ¹⁶ but has yet to be used extensively for matrix element calculations of the kind to be described.

Returning now to the Heavy Quark Effective Theory, in the heavy quark limit, matrix elements involving heavy quarks depend only on the 4-velocity v of the heavy quark and not on the spin or flavour. Apart from radiative corrections arising from matching the effective theory to full QCD, away from the heavy quark limit there will be corrections of order Λ_{QCD}/m_Q . The lattice allows these correction terms to be explored and measured. Particularly for charm quarks, the corrections are potentially large and a non-perturbative study (such as the lattice) is needed to determine them reliably.

3.1. *Heavy \rightarrow heavy transitions*

Semi-leptonic decays such as $\bar{B} \rightarrow D l \bar{\nu}$ are typical applications. In the heavy quark limit, the decays are described by the universal form factor $\xi(\omega)$ (Isgur-Wise form factor) which depends only on the scalar $\omega = v \cdot v'$, where v and v' are the initial and final heavy-quark 4-velocities. At the zero-recoil point ($\omega = 1$), the form factor is normalised by $\xi(1)=1$ up to perturbatively-calculable radiative corrections. The corrections of first order in Λ/m_Q are absent at this point which facilitates comparisons using heavy quarks of finite masses. The HQET has no specific prediction to make of the value of the form factor away from this zero-recoil point. Lattice studies allow this to be determined — see fig 5. Here the shape of the form factor as determined from lattice work ³⁰ is compared with the distribution from experimental data ³¹. The distributions are seen to have shapes which are consistent. Thus the ratio of the hadronic matrix element determined on a lattice to the experimental data can be obtained for a range of ω -values which determines the CKM matrix element $|V_{cb}|$ relatively precisely.

For transitions between baryons each containing a heavy quark, the heavy quark limit will again be given by a universal function. This function will be different from the mesonic case. A study has been made of the baryonic Isgur-Wise function appro-

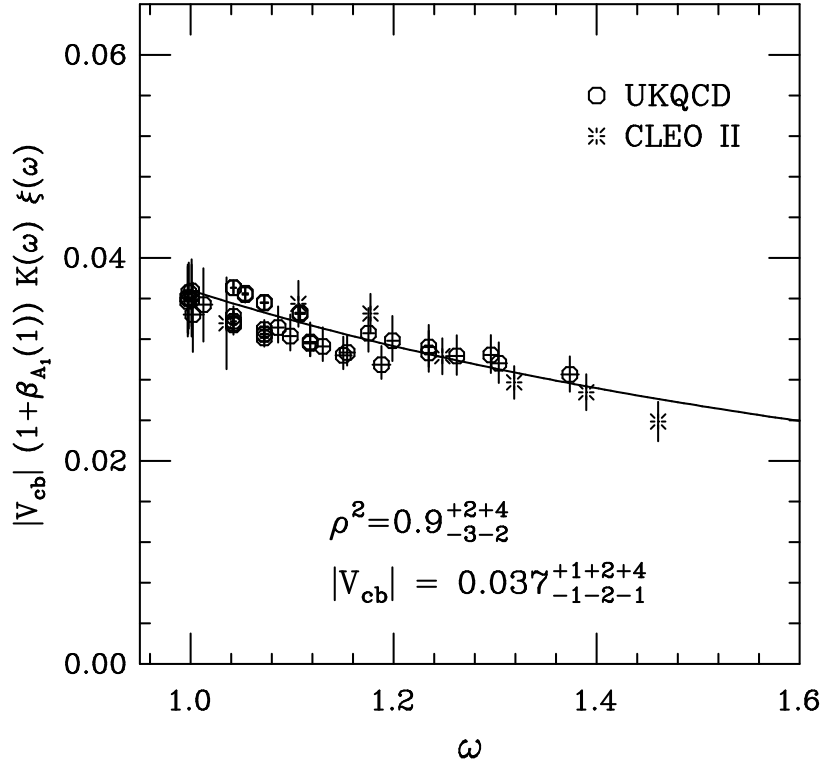


Fig. 5. The variation with $\omega = v.v'$ of the amplitude for the semi-leptonic decay $\bar{B} \rightarrow D^* l \bar{\nu}$. The lattice results³⁰ and experimental data³¹ are compared. They are seen to have similar shapes so that the CKM matrix element $|V_{cb}|K$ can be determined (where K is a multiplicative radiative correction factor of around 0.93).

appropriate for semi-leptonic transitions $\Lambda_Q \rightarrow \Lambda_{Q'}$. A preliminary attempt to determine this on a lattice has recently been made³³. The baryonic form factor shows a faster fall-off with ω than the mesonic case. Applications of this result can be made to semi-leptonic decays such as $\Lambda_b \rightarrow \Lambda_c l \bar{\nu}$.

3.2. Heavy \rightarrow Light transitions

Pseudoscalar decay constants. Pseudoscalar heavy-quark mesons B and D have decay constants defined analogously to f_π and f_K (where we adopt the normalisation convention that $f_\pi = 132$ MeV). Since the direct leptonic decays of B and D are not measured accurately (they have very small branching ratio), a theoretical determination is needed. Again the lattice results can be analysed effectively by using the framework of the HQET. In the heavy quark limit of $m_Q \rightarrow \infty$, the mass M_P of the pseudoscalar meson satisfies $M_P/m_Q \rightarrow 1$. The decay constant for this meson has the behaviour that $f_P \sqrt{M_P}$ is a constant up to radiative corrections and has an

expansion in powers of $1/M_P$ about this heavy quark limit.

Using lattice results from static quarks and propagating quarks, this combination can be determined for a range of values of $1/M_P$. Typical results are shown in fig 6. A consistent description of the results is obtained using a quadratic expression in $1/M_P$. The importance of leading and next-to-leading (in Λ_{QCD}/M_P) corrections to the HQET for this quantity were something of a surprise. Now that the non-perturbative lattice results are well established, the lattice data allow a reliable interpolation to the B meson mass. The D meson case is determined more directly from the propagating quark results alone. The light quark masses used can be varied which enables results for D, D_s , B and B_s to be obtained.

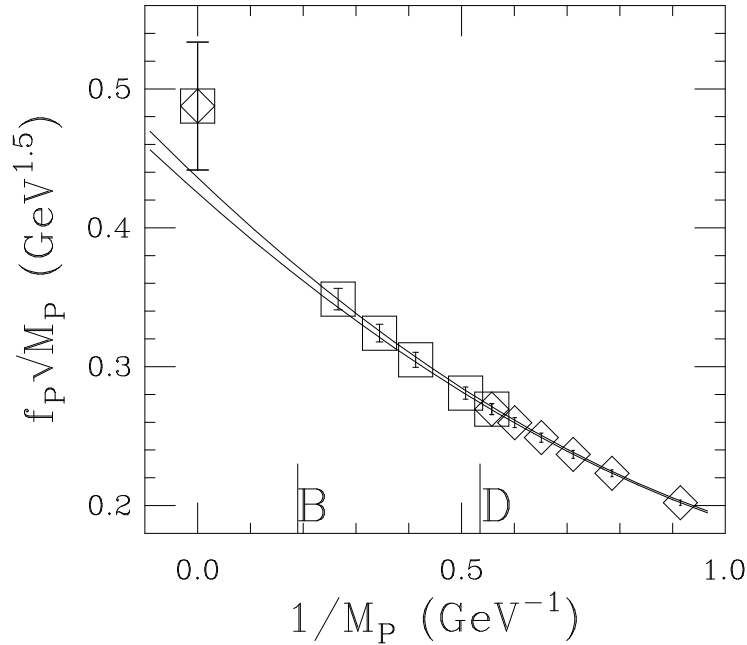


Fig. 6. The pseudoscalar decay constant f_P at different heavy quark masses from quenched lattices ³⁶. The combination $f_P \sqrt{M_P}$ is plotted versus the inverse pseudoscalar meson mass $1/M_P$ since this quantity has a finite limit as $m_Q \rightarrow \infty$ in the Heavy Quark Effective Theory. The curves show quadratic fits to the propagating quark results. These curves extrapolate close to the static value shown. The B and D meson mass values are marked on the scale.

In any lattice determination of matrix elements there are several sources of uncertainty. The continuum limit needs to be obtained by extrapolating ratios of observables to $a = 0$ as was the case for mass values. In quenched studies, there remains an overall scale uncertainty. The new feature is that finite renormalisation factors Z are needed to take account of the differences of the lattice action and the continuum

action. These can be calculated perturbatively. In some cases, however, the first order perturbative estimate for Z gives a relatively large correction so that higher order corrections could be significant.

However, the lattice is a non-perturbative method and it would be preferable to obtain continuum matrix elements without recourse to perturbative approximations. Non-perturbative determination of the relevant Z factors is feasible³⁵. One method makes use of Ward identities and another uses explicit gauge fixing to define quark and gluon Green functions. This is currently an area of active study and a more complete understanding of the systematic errors on estimates of the Z factors will be forthcoming.

From a conservative approach to extracting the continuum limit, Sommer⁸ concluded that the quenched lattice determination of the heavy quark pseudoscalar decay constants is

$$f_D = 176(34) \text{ MeV} \quad f_B = 180(48) \text{ MeV}.$$

The major part of the error quoted comes from the systematic errors (continuum limit, Z factors and scale). An average of more recent lattice results by Allton³⁴ gives $f_D \approx f_B = 200 \pm 20 \text{ MeV}$. The lattice studies are conducted with a range of light quark masses so that the dependence on light quark mass can be determined. The ratios of f_{D_s}/f_D and f_{B_s}/f_B are independent of systematic errors on the Z -factors and the scale and are found to be between 1.1 and 1.2 in most lattice studies²⁸.

As yet there have been only limited studies going beyond the quenched approximation. One recent result³⁶ obtains a value for f_B some 40 MeV higher than the quenched value at a similar lattice spacing. Since the approach to the continuum limit could be different for quenched and full QCD, this full QCD result at a relatively coarse lattice spacing is not necessarily in contradiction with the quenched continuum result. A further study of quenched QCD at several lattice spacings is needed to obtain definitive results with all systematic errors under control.

$B\bar{B}$ mixing. The B -parameter describes the hadronic four-quark matrix element appropriate to $B\bar{B}$ mixing. This can be determined from lattice studies. A recent calculation³⁷ using the static approximation obtained

$$B_B = 0.99(6)(3) \quad \text{and} \quad B_{B_s} = 1.01(5)(3)$$

where the second error is the systematic error coming from the scale. There are also additional systematic errors associated with the matching coefficients (Z -factors), with extrapolation from the static limit of infinite quark mass to the B mass (ie corrections of order Λ_{QCD}/M_B) and with quenching. The lattice results for B_B are close to the naive value of unity for this matrix element.

The combination $f_B^2 B_B$ enters $B\bar{B}$ mixing. Since the errors on f_B coming from the Z factors and scale are relatively large, it is useful to take the ratio between strange

and non-strange B mesons to have a better determined quantity. Thus

$$\frac{f_{B_s}^2 B_{B_s} M_{B_s}}{f_{B_d}^2 B_{B_d} M_{B_d}} = 1.37(9)(5)$$

from a recent quenched lattice evaluation ³⁷ in the static limit. This ratio allows the observed $B\bar{B}$ mixing coefficients $\Delta m_s/\Delta m_d$ to be related to the CKM matrix element ratio $(V_{ts}/V_{td})^2$. The present experimental situation ³⁸ is that only a lower bound exists on Δm_s (namely $\Delta m_s > 9\tau_{B_s}$) and this gives constraints on the CKM matrix elements which are consistent with other information.

Conversely, one can use existing knowledge of CKM matrix elements and the lattice estimates of the hadronic matrix elements to evaluate a prediction for $x_s = \Delta m_s/\tau_{B_s}$. Assuming $B_K = 0.8$ and $f_B = 180$ MeV, a rather large value of $x_s \approx 16$ is indicated ³⁷ and such a large value would render mixing effects for B_s very hard to detect experimentally.

Rare B decays. Decays such as $B \rightarrow K^*\gamma$ are sensitive tests of the Standard Model since they are forbidden at tree-level (as a flavour changing neutral current) and only proceed via loops with heavy intermediate states (W and t). One of the relevant contributions comes from the penguin diagram. The short-distance part can be calculated perturbatively but the hadronic matrix element between B and K^* and the appropriate four-quark operator is needed. This can be calculated in principle by lattice methods. At present this is a rather exploratory study and results are not yet fully established. Nevertheless, as an example of the potential of the lattice technique, I discuss the possibilities.

The appropriate hadronic matrix element can be calculated using propagating heavy quarks on the lattice. This does not allow to reach the B-meson mass without an extrapolation. For an improved fermionic action such as the SW-clover action, the range of mass values (or momentum values) of the heavy quark that can be reliably used is larger.

In order to establish a theoretical framework for the extrapolation in heavy quark mass, one can use the HQET which implies that physics is similar at a fixed value of

$$\omega = v.v' = 1 + \frac{(M - m)^2 - q^2}{2Mm}$$

where M is the heavy Meson mass and m the K^* mass. For the decay $B \rightarrow K^*\gamma$, the photon is on-shell with $q^2 = 0$ and thus $\omega = 3.04$. This large value of ω implies, at a lower value of m_Q , a large value of q^2 . Present lattice studies are not able to explore this region of ω directly. Models for the q^2 dependence of the matrix element can be constructed and checked against measurements in the available q^2 interval. A range of plausible extrapolations can be found ³⁹, but they lead to a rather wide band of predictions for the decay rate. One way to focus on the hadronic matrix element is to study the branching ratio to the inclusive radiative decay to strange mesons

$(B \rightarrow K^* \gamma)/(b \rightarrow s \gamma)$. This branching ratio is $19 \pm 6 \pm 4$ % experimentally ³² and values straddling this are obtained ³⁹ from different extrapolation assumptions from lattice data.

Further progress needs a reduced lattice spacing to allow larger quark masses and/or large momenta to be explored directly. The HQET relates the hadronic matrix elements for the rare decay $B \rightarrow K^* \gamma$ to those for the semi-leptonic decays $B \rightarrow Ll\bar{\nu}$ and $D \rightarrow Ll\bar{\nu}$ where L stands for a light meson: π , ρ , K or K^* . Some theoretical input can also come from a combined study of these related processes. Here I stress one such application.

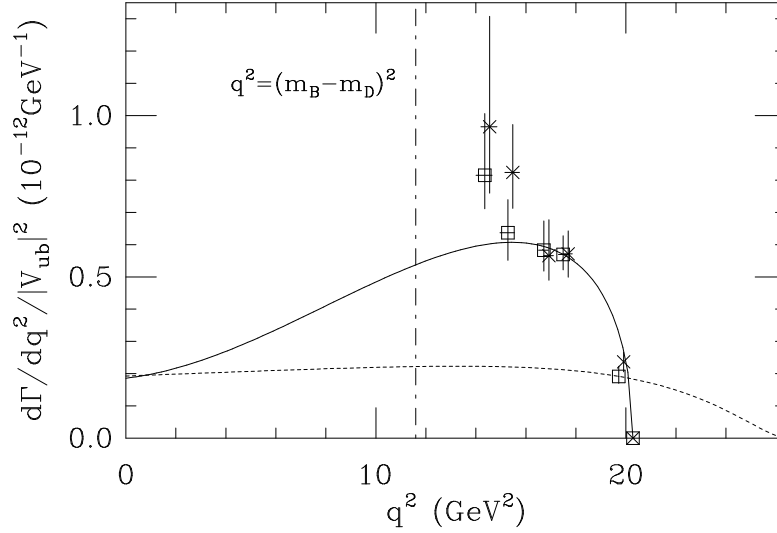


Fig. 7. The distribution versus q^2 of the rate for the decay $\bar{B}^0 \rightarrow \rho^+ l^- \bar{\nu}$. The lattice results ⁴⁰ for the hadronic matrix element are shown together with a continuous curve which is a plausible estimate of the expected shape of the distribution. The vertical dash-dotted line represents the upper limit in q^2 for charm production. The dotted curve illustrates a plausible shape for the decay $\bar{B}^0 \rightarrow \pi^+ l^- \bar{\nu}$.

$B \rightarrow \rho + l + \bar{\nu}$. Exclusive semi-leptonic decays of B mesons to non-strange light hadrons give a route to determine the V_{ub} CKM mixing strengths. As well as accurate experimental data on branching ratios, the hadronic matrix element needs to be evaluated from QCD accurately to allow this determination. This has recently been addressed from lattice measurements.

Currently one can study on the lattice the relevant matrix elements in the region $m_Q \approx m_c$ and $q^2 \lesssim m_c^2$. Using again HQET to control the extrapolation in m_Q to m_b , implies that the region explored on the lattice corresponds⁴⁰ for B decay to the region of phase space where the momentum transfer q from the heavy quark (Q) to the light meson (ρ) is near its maximum. This is illustrated in fig 7. This is a phase space region which is expected to contribute to a substantial fraction of the decay events. Moreover, this region is above the endpoint for charm production in the decay and thus there will be no contamination from the larger branching fraction to charm final states. As experimental data on this exclusive decay becomes more precise, the comparison of experimental and lattice distributions in this interval at large q^2 will provide an accurate method to determine $|V_{ub}|$.

4. Conclusions

The lattice technique is now a well established and quantitative method to obtain *continuum* results in quenched QCD. I have discussed applications to glueballs, mesons and determination of α_s . Other topics, only partly discussed here, include mesons and baryons involving one heavy quark, hybrid mesons and the $c\bar{c}$ and $b\bar{b}$ systems. As well as masses, decay matrix elements, form factors and moments of structure functions can be evaluated. Another important area of study which relies heavily on lattice input is hadronic physics at non-zero temperature. There are limitations to the current lattice capabilities: only quantities which can be expressed as vacuum expectation values of fields can be explored easily. Consequently, for example, (i) non-zero chemical potential is hard to explore, and (ii) scattering can only be studied indirectly and in a limited regime.

The bigger limitation is the quenched approximation itself. No comprehensive studies have been made of the corrections caused by quenching. Most of our experience comes from results for two flavours of sea quarks that are too heavy and with a rather coarse lattice spacing. This situation can be improved either by increased computing resources or by improved algorithms. The lattice community is working hard on both fronts. At present 10 – 50 Gflops-year is the typical resource available to Grand Challenge lattice projects (this is $\approx 10^{18}$ floating point operations). There are well-advanced plans for computing resources providing 100 – 500 Gflops dedicated to lattice work. The increase in computational power will allow fuller studies of the systematic errors coming from quenching.

Meanwhile, a way to explore beyond the quenched approximation is to evaluate explicit quark loops in the quenched approximation. Several groups have explored this route and have been able to address topics such as the spin content of the nucleon, the η – η' system and hadronic decay widths. Estimates of hadronic scattering lengths have also been given.

A fuller understanding of the determination of hadronic matrix elements is in

progress. This is important for the particle physics community as a whole, since accurate values of such matrix elements are an essential component of precision tests of the Standard Model and searches for physics beyond the Standard Model.

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